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DUALITY FOR PARTIAL GROUP ACTIONS

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Abstract. Given a finite group G acting as automorphisms on a ring $\mathcal{A}$, the skew group ring $\mathcal{A} * G$ is an important tool for studying the structure of G-stable ideals of $\mathcal{A}$. The ring $\mathcal{A} * G$ is G-graded, i.e. G coacts on $\mathcal{A} * G$. The Cohen-Montgomery duality says that the smash product $\mathcal{A}*G\#k[G]^*$ of $\mathcal{A}*G$ with the dual group ring $k[G]^*$ is isomorphic to the full matrix ring $M_n(\mathcal{A})$ over $\mathcal{A}$, where n is the order of G. In this note we show how much of the Cohen-Montgomery duality carries over to partial group actions in the sense of R.Exel. In particular we show that the smash product $(\mathcal{A} *_\alpha G)\,\#k[G]^*$ of the partial skew group ring $\mathcal{A}*_\alpha G$ and $k[G]^*$ is isomorphic to a direct product of the form $K \times \mathbf{e}M_n(\mathcal{A})\mathbf{e}$ where $\mathbf{e}$ is a certain idempotent of $M_n(\mathcal{A})$ and K is a subalgebra of $(\mathcal{A} *_\alpha G)\,\#k[G]^*$. Moreover $\mathcal{A}*_\alpha G$ is shown to be isomorphic to a separable subalgebra of $\mathbf{e}M_n(\mathcal{A})\mathbf{e}$. We also look at duality for infinite partial group actions.

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1. Introduction

Let k be a commutative unital ring and $\mathcal{A}$ a untial k-algebra. Given a finite group G acting as k-linear automorphisms on $\mathcal{A}$, Cohen and Montgomery showed in [1] that the smash product $\mathcal{A} * G\#k[G]^*$ of the skew group ring $\mathcal{A} * G$ and the dual group ring $k[G]^* = \mathrm{Hom}(k[G], k)$ is isomorphic to the full matrix ring $M_n(\mathcal{A})$ over $\mathcal{A}$, where n is the order of G.

The notion of a partial group action on a k-algebra $\mathcal{A}$ has been introduced by R.Exel in the study of C^*-algebras (see [4]). One says that G *acts partially on* $\mathcal{A}$

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by a family $\{\alpha_g : D_{g^{-1}} \to D_g\}_{g\in G}$ if for all $g \in G$, D_g is an ideal of $\mathcal{A}$ and α_g is an isomorphism of k-algebras such that for all $g, h \in G$:

(i) $D_e = \mathcal{A}$ and α_e is the identity map of $\mathcal{A}$;

(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;

(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for all $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

The partial skew group ring of $\mathcal{A}$ and G is defined to be the projective left $\mathcal{A}$-module $\mathcal{A} *_\alpha G = \bigoplus_{g\in G} D_g$ whose multiplication will be defined in the next section. Since $\mathcal{A} *_\alpha G$ is naturally G-graded, the question arises how much of the Cohen-Montgomery duality carries over to partial group actions.

As in [3] we will assume that the ideals D_g are generated by central idempotents, i.e. $D_g = \mathcal{A}1_g$ with central idempotent $1_g \in \mathcal{A}$ for all $g \in G$. For any $g \in G$ we define the following endomorphism $\beta_g : A \to A$ of $\mathcal{A}$ by

$$\beta_g(a) = \alpha_g(a1_{g^{-1}}) \quad \forall a \in \mathcal{A}$$

This map gives rise to a k-linear map $k[G] \otimes A \to A$ with

$$g \otimes a \mapsto g \cdot a := \beta_g(a) = \alpha_g(a1_{g^{-1}})$$

for all $g \in G, a \in \mathcal{A}$.

Lemma 1.1. *With the notation above we have that*

(1) *β_g are k-algebra endomorphisms of $\mathcal{A}$ for all $g \in G$, i.e.*

$$g \cdot (ab) = (g \cdot a)(g \cdot b) \quad \forall a, b \in \mathcal{A}.$$

(2) *$g \cdot (h \cdot a) = ((gh) \cdot a)1_g$ for all $g, h \in G$ and $a \in \mathcal{A}$.*

(3) *$(g \cdot a)b = g \cdot (a(g^{-1} \cdot b))$ for all $a, b \in \mathcal{A}$ and $g \in G$.*

Proof. (1) follows since the α_g are algebra homomorphisms and the idempotents 1_g are central, i.e. for all $a, b \in \mathcal{A}$:

$$\beta_g(ab) = \alpha_g(ab1_{g^{-1}}) = \alpha_g(a1_{g^{-1}}b1_{g^{-1}}) = \alpha_g(a1_{g^{-1}})\alpha_g(b1_{g^{-1}}) = \beta_g(a)\beta_g(b).$$

(2) follows from [3, 2.1(ii)]:

$$\alpha_g(\alpha_h(a1_{h^{-1}})1_{g^{-1}}) = \alpha_{gh}(a1_{h^{-1}g^{-1}})1_g$$

what expressed by β yields the statement of (2).

(3) Using (1), (2) and the fact that $\beta_e = id$ and that the image of β_g is $D_g = A1_g$ we have that

$$g \cdot (a(g^{-1} \cdot b)) = (g \cdot a)(g \cdot (g^{-1} \cdot b)) = (g \cdot a)b1_g = (g \cdot a)b.$$

□

Obviously we also have $g\cdot 1 = \alpha_g(1_{g^{-1}}) = 1_g$ and $g\cdot(g^{-1}\cdot a) = ((gg^{-1})\cdot a)1_g = a1_g$ for all $a \in \mathcal{A}$ and $g \in G$ using property (2). Moreover using the fact that α_g is bijective and 1_g central we have for all $a \in \mathcal{A}$ and $g \in G$ that $g \cdot a = 0$ if and only if $a \in \mathcal{A}(1 - 1_g)$.

2. Grading of the partial skew group ring

The partial skew group ring is the projective left $\mathcal{A}$-module $\mathcal{A} *_\alpha G = \bigoplus_{g\in G} D_g$. We will write an element of $\mathcal{A} *_\alpha G$ as a finite sum of elements $\sum_{g\in G} a_g\overline{g}$ where $a_g \in D_g = \mathcal{A}1_g$ and $\overline{g}$ is a placeholder for the g-th component. $\mathcal{A} *_\alpha G$ becomes an associative k-algebra by the product:

$$(a\overline{g})(b\overline{h}) = \alpha_g(\alpha_{g^{-1}}(a)b)\overline{gh}$$

for all $g, h \in G$ and $a \in D_g$ and $b \in D_h$. Using our "$\cdot$"-notation we see easily

$$(a\overline{g})(b\overline{h}) = a(g \cdot b)\overline{gh}.$$

The algebra $\mathcal{A} *_\alpha G$ is naturally G-graded where the homogeneous elements are those in $\{D_g\}_{g\in G}$, i.e. $D_gD_h \subseteq D_{gh}$ by definition of the multiplication in $\mathcal{A} *_\alpha G$. Thus $\mathcal{A} *_\alpha G$ becomes a $k[G]$-comodule algebra. Note that the G-grading is strong, in the sense that $D_gD_h = D_{gh}$ if and only if $D_g = \mathcal{A}$ for all $g \in G$, i.e. the G-action is global (since if $D_gD_h = D_{gh}$ for all $g, h \in G$, then

$$\mathcal{A}1_g1_{g^{-1}} = D_gD_{g^{-1}} = D_{gg^{-1}} = D_e = \mathcal{A},$$

thus 1_g is an invertible central idempotent and hence equals 1, i.e. $D_g = \mathcal{A}$). Known results on graded rings can be applied to the G-grading of $\mathcal{A} *_\alpha G$. and we will point out some of those results now. Recall that a graded ring is called graded semiprime, if it has no non-zero nilpotent graded ideals.

Theorem 2.1. *Let G be a finite group acting partially on $\mathcal{A}$.*

(1) *$\mathcal{A}$ is semiprime if and only if $\mathcal{A} *_\alpha G$ is graded semiprime.*
(2) *If $\mathcal{A}$ is $|G|$-torsion free, then $\mathcal{A}$ is semiprime if and only if $\mathcal{A} *_\alpha G$ is semiprime.*
(3) *If $P \subsetneq Q$ are prime ideals in $\mathcal{A} *_\alpha G$, then $P \cap A \subsetneq Q \cap A$ are primes in $\mathcal{A}$.*
(4) *If P is a prime in $\mathcal{A} *_\alpha G$, then there are $k \leq |G|$ primes $p_1, \ldots, p_k$ in $\mathcal{A}$ minimal over $P\cap A$, and moreover $P\cap A = p_1\cap\cdots\cap p_k$. The set $\{p_1, \ldots, p_k\}$ is uniquely determined by P.*
(5) *Given any prime p of $\mathcal{A}$, there exists a prime P of $\mathcal{A} *_\alpha G$ so that p is minimal over $P \cap A$. There are at most $m \leq |G|$ such primes $P_1, \ldots, P_m$ of $\mathcal{A} *_\alpha G$.*

Proof. (1) follows from [1, 2.9], if we show that the grading of the partial skew group ring is *non-degenerated*. The grading of a G-graded ring $\mathcal{A} = \bigoplus_{g\in G} A_g$ is called *non-degenerated* if for any $g \in G$ and $0 \neq a_g \in A_g$ also $a_g A_{g^{-1}} \neq 0 \neq A_{g^{-1}} a_g$ (see [1, Lemma 2.5]). Take any $0 \neq a_g = a\overline{g} \in A_g = D_g\overline{g}$ of the partial skew group ring $\mathcal{A} *_\alpha G$. Then

$$0 \neq a\overline{e} = (a\overline{g})\left(1_{g^{-1}}\overline{g}^{-1}\right) \in a_g A_{g^{-1}} \quad \text{and}$$

$$0 \neq \alpha_g^{-1}(a)\overline{e} = 1_{g^{-1}}\left(g^{-1}\cdot a\right)\overline{e} = \left(1_{g^{-1}}\overline{g}^{-1}\right)(a\overline{g}) \in A_{g^{-1}} a_g.$$

Hence the G-grading of $\mathcal{A} *_\alpha G$ is non-degenerated.

(2) follows from [1, 5.5]; (3) follows from [1, 7.1]; (4)+(5) follow from [1, 7.3]. □

3. Duality for partial actions of finite groups

Assume G to be finite, then $k[G]^*$ becomes a Hopf algebra with projective basis $p_g \in k[G]^*$ where $p_g(h) = \delta_{g,h}$ for all $g, h \in H$. The multiplication is defined as $p_g * p_h = \delta_{g,h} p_g$ and the identity element of $k[G]^*$ is $1 = \sum_{h\in H} p_h$. Now $\mathcal{A} *_\alpha G$ becomes a $k[G]^*$-module algebra by

$$p_h \triangleright (a\overline{g}) = \delta_{g,h} a\overline{g}$$

for all $g, h \in G$ and $a_g \in D_g$. The multiplication of the smash product $(\mathcal{A} *_\alpha G)\#k[G]^*$ is defined as

$$(a\overline{g}\#p_h)(b\overline{k}\#p_l) = \sum_{s\in G}(a\overline{g})[p_s\triangleright(b\overline{k})]\#p_{s^{-1}h}*p_l = (a\overline{g})(b\overline{k})\#p_{k^{-1}h}*p_l = a(g\cdot b)\overline{gk}\#\delta_{h,kl}p_l.$$

The identity element of $\mathcal{B} = \mathcal{A} *_\alpha G\#k[G]^*$ is $\sum_{h\in G} 1\overline{e}\#p_h$. In the case of global actions Cohen and Montgomery proved in [1] that $\mathcal{A} * G\#k[G]^* \simeq M_n(\mathcal{A})$ where $n = |G|$ and $M_n(\mathcal{A})$ denotes the ring of $n\times n$-matrices over $\mathcal{A}$. We will index the matrices of $M_n(\mathcal{A})$ by elements of G and denote by $E_{g,h}$ the elementary matrix that has the value 1 in the g-th row and the h-th column and zero elsewhere.

Proposition 3.1. *Let G be a finite group of n elements, acting partially on a k-algebra $\mathcal{A}$ and consider the k-algebra $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^*$. The map*

$$\Phi : \mathcal{B} \longrightarrow M_n(\mathcal{A}) \quad \textit{with}$$

$$\sum_{g,h} a_{g,h}\overline{g}\#p_h \mapsto \sum_{g,h} h^{-1}\cdot(g^{-1}\cdot a_{g,h})E_{gh,h}$$

is a k-algebra homomorphism.

Proof. First note that for any $g, h, k \in G$ and $a \in D_g, b \in D_h$ we have, using Lemma refproperties(2) in the 2nd, 4th and 6th line and Lemma 1.1(1) in the 3rd line:

$$\begin{aligned} k^{-1} \cdot ((gh)^{-1} \cdot (a(g \cdot b))) &= k^{-1} \cdot \left(((gh)^{-1} \cdot a)((gh)^{-1} \cdot (g \cdot b))\right) \\ &= \left[k^{-1} \cdot ((gh)^{-1} \cdot a)\right] \left[k^{-1} \cdot (h^{-1} \cdot b)\right] \\ &= ((ghk)^{-1} \cdot a)((hk)^{-1} \cdot b)1_{k^{-1}} \\ &= ((ghk)^{-1} \cdot a)1_{(hk)^{-1}}((hk)^{-1} \cdot b)1_{k^{-1}} \\ &= ((hk)^{-1} \cdot (g^{-1} \cdot a))(k^{-1} \cdot (h^{-1} \cdot b)) \end{aligned}$$

Thus we showed:

$$k^{-1} \cdot ((gh)^{-1} \cdot (a(g \cdot b))) = ((hk)^{-1} \cdot (g^{-1} \cdot a))(k^{-1} \cdot (h^{-1} \cdot b)) \tag{1}$$

For any $a\overline{g}\#p_h, b\overline{k}\#p_l \in (\mathcal{A} *_\alpha G) \# k[G]^*$ we have, using equation (1):

$$\begin{aligned} \Phi((a\overline{g}\#p_h)(b\overline{k}\#p_l)) &= \Phi(a(g \cdot b)\overline{gk}\#\delta_{h,kl}p_l) \\ &= l^{-1} \cdot ((gk)^{-1} \cdot (a(g \cdot b)))E_{gkl,l}\delta_{h,kl} \\ &= ((kl)^{-1} \cdot (g^{-1} \cdot a))(l^{-1} \cdot (k^{-1} \cdot b))E_{gh,h}E_{kl,l}\delta_{h,kl} \\ &= (h^{-1} \cdot (g^{-1} \cdot a))E_{gh,h}\,(l^{-1} \cdot (k^{-1} \cdot b))E_{kl,l} \\ &= \Phi(a\overline{g}\#p_h)\Phi(b\overline{k}\#p_l) \end{aligned}$$

Hence Φ is an algebra homomorphism. □

Note that Φ restricted to $\mathcal{A} *_\alpha G$ is injective, i.e. $\mathcal{A} *_\alpha G$ can be considered a subalgebra of $M_n(\mathcal{A})$. In general $\mathrm{Ker}(\Phi)$ is non-trivial, unless the partial action is a global action. Recall the partial order on the boolean algebra $B(\mathcal{A})$ of central idempotents of $\mathcal{A}$: for any $e, f \in B(\mathcal{A}) : e \leq f \Leftrightarrow e = ef$. For our situation of a partial group action G on $\mathcal{A}$ set for any $g \in G$:

$$\Lambda_g = \{h \in G \mid 1_g \not\leq 1_{gh}\}$$

Proposition 3.2. $\mathrm{Ker}(\Phi) = \bigoplus_{g\in G} \bigoplus_{h\in\Lambda_g} \mathcal{A}(1 - 1_{gh})1_g\overline{g}\#p_h$.

Proof. Suppose $\gamma = \sum_{g,h} a_{g,h}\overline{g}\#p_h \in \mathrm{Ker}(\Phi)$, then $h^{-1} \cdot (g^{-1} \cdot a_{g,h}) = 0$ for all $g, h \in G$. Thus $(g^{-1} \cdot a_{g,h}) \in \mathcal{A}(1 - 1_h) \cap D_{g^{-1}} = \mathcal{A}(1 - 1_h)1_{g^{-1}}$. Hence

$$a_{g,h} = g \cdot (g^{-1} \cdot a_{g,h}) \in \mathcal{A}g \cdot (1 - 1_h) = \mathcal{A}(1_g - 1_g1_{gh}),$$

i.e. $\gamma \in \bigoplus_{g,h} \mathcal{A}(1 - 1_{gh})1_g\overline{g}\#p_h = \bigoplus_{g\in G} \bigoplus_{h\in\Lambda_g} \mathcal{A}(1 - 1_{gh})1_g\overline{g}\#p_h$. The other inclusion follows because $\Phi\left((g \cdot (1 - 1_h))\overline{g}\#p_h\right) = h^{-1} \cdot (g^{-1} \cdot (g \cdot (1 - 1_h)))E_{gh,h} = h^{-1} \cdot ((1 - 1_h)1_g)E_{gh,h} = 0$. □

Hence the kernel depends on the partial order of the central idempotent 1_g. In particular $\Lambda_e = \emptyset$ means $1 = 1_g$ for all $g \in G$.

Note that the inclusion of $\mathcal{A} *_\alpha G$ into $(\mathcal{A} *_\alpha G)\#k[G]^*$ is given by $a\overline{g} \mapsto \sum_{h\in G} a\overline{g}\#p_h$ for all $g \in G$ and $a \in D_g$. If $\sum_{h\in G} a\overline{g}\#p_h \in \mathrm{Ker}(\Phi)$, then $a \in \mathcal{A}(1-1_{gh})1_g$ for all $h \in G$. In particular for $h = e$ we have $a \in \mathcal{A}(1-1_g)1_g = 0$. Hence Φ restricted to $\mathcal{A} *_\alpha G$ is injective.

We will describe the image of Φ. By definition of Φ, the image of an arbitrary element $\gamma = \sum_{g,h} a_{g,h}\overline{g}\#p_h$ is

$$\Phi(\gamma) = \sum_{g,h} ((gh)^{-1} \cdot a_{g,h}) 1_{(gh)^{-1}} 1_{h^{-1}} E_{gh,h} = (b_{r,s} 1_{r^{-1}} 1_{s^{-1}})_{r,s\in G}$$

with $b_{r,s} = r^{-1} \cdot a_{rs^{-1},s}$ for all $r, s \in G$.

Proposition 3.3. *The image of Φ consists of all matrices of the form $\left(b_{g,h} 1_{g^{-1}} 1_{h^{-1}}\right)_{g,h\in G}$ for any matrix $(b_{g,h})$ of elements of $\mathcal{A}$. In particular $\mathrm{Im}(\Phi) = \mathbf{e}M_n(A)\mathbf{e}$, where $\mathbf{e}$ is the idempotent $\sum_{g\in G} 1_{g^{-1}} E_{g,g}$.*

Proof. We saw already that an element of the image of Φ is of the given form. Note that by definition of partial group action we have

$$D_g \cap D_{gh} = \alpha_g(D_{g^{-1}} \cap D_h)$$

for all $g, h \in G$. Hence also

$$D_{g^{-1}} \cap D_{h^{-1}} = \alpha_{g^{-1}}(D_g \cap D_{gh^{-1}})$$

holds for all $g, h \in G$. Thus for all $b \in \mathcal{A}$ there exists $a \in \mathcal{A}$ such that

$$b 1_{g^{-1}} 1_{h^{-1}} = \alpha_{g^{-1}}(a 1_{gh^{-1}} 1_g) = g^{-1} \cdot (a 1_{gh^{-1}}).$$

This implies that

$$\begin{aligned}
\Phi(a 1_g 1_{gh^{-1}} \overline{gh^{-1}}\#p_h) &= h^{-1} \cdot ((hg^{-1}) \cdot (a 1_g 1_{gh^{-1}})) E_{g,h} \\
&= g^{-1} \cdot (a 1_g 1_{gh^{-1}})) 1_{h^{-1}} E_{g,h} \\
&= b 1_{g^{-1}} 1_{h^{-1}} E_{g,h}
\end{aligned}$$

Hence given any matrix $(b_{g,h})$ there are elements $a_{g,h}$ such that

$$\Phi\left(\sum_{g,h} a_{g,h} 1_g 1_{gh^{-1}} \overline{gh^{-1}}\#p_h\right) = \sum_{g,h} b_{g,h} 1_{g^{-1}} 1_{h^{-1}} E_{g,h} = \left(b_{g,h} 1_{g^{-1}} 1_{h^{-1}}\right)_{g,h\in G}.$$

This shows that $\mathrm{Im}(\Phi)$ consists of all matrices of the given form and hence is equal to $\mathbf{e}M_n(A)\mathbf{e}$. Note that $\mathbf{e}$ is the image of the identity element of $\mathcal{B}$. □

The last Propositions yield our main result in this section

Theorem 3.4. $(\mathcal{A} *_\alpha G)\#k[G]^* \simeq \operatorname{Ker}(\Phi) \times \mathbf{e}M_n(\mathcal{A})\mathbf{e}$.

Proof. The kernel of Φ is an ideal and a direct summand of $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^*$. To see this we first show that the left $\mathcal{A}$-module $I = \bigoplus_{g,h\in G} \mathcal{A}1_{gh}1_g\overline{g}\#p_h$ is a two-sided ideal of $\mathcal{B}$. For any $x\overline{k}\#p_l \in \mathcal{B}$ and $a1_{gh}1_g\overline{g}\#p_h \in I$ we have

$$\begin{aligned}(a1_{gh}1_g\overline{g}\#p_h)(b\overline{k}\#p_l) &= a1_{gh}1_g(g\cdot b1_k)\overline{gk}\#\delta_{h,kl}p_l = a(g\cdot b)\delta_{h,kl}1_{gkl}1_{gk}\overline{gk}\#p_l \in I.\\ (b\overline{k}\#p_l)(a1_{gh}1_g\overline{g}\#p_h) &= b(k\cdot a1_{gh}1_g)\overline{kg}\#\delta_{k,gh}p_h = b(g\cdot a)\delta_{h,kl}1_{kgh}1_{kg}\overline{kg}\#p_h \in I.\end{aligned}$$

Since $I \oplus \operatorname{Ker}(\Phi) = \mathcal{B}$ and both direct summands are two-sided ideals we have $\mathcal{B} = I \times \operatorname{Ker}(\Phi)$ (ring direct product). Moreover $\Phi(I) = \mathbf{e}M_n(\mathcal{A})\mathbf{e} = \operatorname{Im}(\Phi)$. This implies $\mathcal{B} \simeq \operatorname{Ker}(\Phi) \times \mathbf{e}M_n(\mathcal{A})\mathbf{e}$. □

Note that Φ embedds $\mathcal{A} *_\alpha G$ into the Pierce corner $\mathbf{e}M_n(\mathcal{A})\mathbf{e}$.

Corollary 3.5. *$\mathcal{A} *_\alpha G$ is isomorphic to a separable subalgebra of $\mathbf{e}M_n(\mathcal{A})\mathbf{e}$.*

Proof. Recall that the subalgebra $\mathcal{A} *_\alpha G$ sits into $\mathcal{B}$ by $a\overline{g} \mapsto \sum_{h\in G} a\overline{g}\#p_h$. The right action of $\mathcal{A} *_\alpha G$ on $\mathcal{B}$ is given by

$$(x\overline{k}\#p_l)\cdot a\overline{g} = (x\overline{k}\#p_l)\left(\sum_{h\in G} a\overline{g}\#p_h\right) = (x\overline{k})(a\overline{g})\#p_{g^{-1}l}$$

The left action is given by

$$a\overline{g}\cdot(x\overline{k}\#p_l) = \left(\sum_{h\in G} a\overline{g}\#p_h\right)(x\overline{k}\#p_l) = (a\overline{g})(x\overline{k})\#p_l$$

The element

$$f = \sum_{g\in G} \overline{e}\#p_g \otimes \overline{e}\#p_g \in \mathcal{B} \otimes_{\mathcal{A}*_\alpha G} \mathcal{B}$$

is $\mathcal{A} *_\alpha G$-centralising, i.e. for all $a\overline{h} \in \mathcal{A} *_\alpha G$ we have

$$fa\overline{h} = \sum_{g\in G} \overline{e}\#p_g \otimes a\overline{h}\#p_{h^{-1}g} = \sum_{g\in G} a\overline{h}\#p_{h^{-1}g} \otimes \overline{e}\#p_{h^{-1}g} = a\overline{h}f$$

Since also $\mu(f) = \overline{e}\#\sum_{g\in G} p_g = 1_{\mathcal{B}}$ we have that f is a seperability idempotent for $\mathcal{B}$ over $\mathcal{A} *_\alpha G$. Hence $\mathbf{e}M_n(\mathcal{A})\mathbf{e} \simeq \Phi(\mathcal{B})$ is separable over $\Phi(\mathcal{A} *_\alpha G) \simeq \mathcal{A} *_\alpha G$. □

4. Trivial partial actions

The easiest example of partial actions arise from (central) idempotents in a k-algebra $\mathcal{A}$. Suppose that $\mathcal{A}$ admits a non-zero central idempotent, i.e. there exist algebras R, S such that $\mathcal{A} = R \times S$ as algebras. For any group G set $D_g = R \times 0$ and $\alpha_g = id_{D_g}$for all $g \neq e$ and $D_e = \mathcal{A}$ and $\alpha_e = id_{\mathcal{A}}$. Then $\{\alpha_g \mid g \in G\}$ is a partial action of G on $\mathcal{A}$. The partial skew group ring turns out to be $\mathcal{A} *_\alpha G \simeq R[G] \times S$, where $R[G]$ denotes the group ring of R and G. Note that $0 \times S$ is in the zero-component of the G-grading on $\mathcal{A} *_\alpha G$. If G is finite, say of order n, then a short calculation (using Cohen-Montgomery duality, Proposition 3.2 and Theorem 3.4) shows that $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^*$ is isomorphic to $M_n(R) \times S^n$ where S^n denotes the direct product of n copies of S. Depending on the rings R and S, $\mathcal{B}$ might or might not be Morita equivalent to $\mathcal{A}$. For instance if $R = S = k$ is a field, then any progenerator P for $\mathcal{A}$ has the form $k^r \times k^s$ for numbers $r, s \geq 1$. Thus $\mathrm{End}_k(P) \simeq M_r(k) \times M_s(k)$, whose center is isomorphic to $k^2 = \mathcal{A}$. On the other hand $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^* \simeq M_n(k) \times k^n$ has center k^{n+1}, i.e. $\mathcal{B}$ will be Morita equivalent to $\mathcal{A}$ if and only if G is trivial.

On the other hand, there are algebras which satisfy (as algebras) $\mathcal{A}^n \simeq \mathcal{A} \simeq M_n(\mathcal{A})$ for any n. To give an example, let R be the ring of sequences of elements of a field k, i.e. $R = k^{\mathbb{N}}$ with componentwise multiplication and addition. The function $\mathbf{e}$ with $\mathbf{e}(2n) = 1$ and $\mathbf{e}(2n+1) = 0$ for all n defines an idempotent of R. The map $\Psi : \mathbf{e}R \to R$ with $\Psi(\mathbf{e}f)(n) = f(2n)$ is a ring isomorphism. Analogously we can show that $(1-\mathbf{e})R \simeq R$. Hence $R^2 \simeq R$. Now take $\mathcal{A} = \mathrm{End}_k(S)$, where $S = R^{(\mathbb{N})}$ denotes the countable infinite free R-module. Using again $\mathbf{e}$ we have that

$$\mathbf{e}\mathcal{A} \simeq (1-\mathbf{e})\mathcal{A} \simeq \mathcal{A} = (\mathbf{e}\mathcal{A}) \times ((1-\mathbf{e})\mathcal{A}) \simeq \mathcal{A} \times \mathcal{A} \simeq \cdots \simeq \mathcal{A}^n$$

for any $n \geq 2$. Moreover for any partition of $\mathbb{N}$ into n infinite disjoint subsets $\Lambda_1, \ldots, \Lambda_n$, we have that

$$S = R^{(\mathbb{N})} \simeq R^{(\Lambda_1)} \oplus \cdots \oplus R^{(\Lambda_n)} \simeq S^n.$$

Hence $\mathcal{A} = \mathrm{End}_k(S) \simeq \mathrm{End}_k(S^n) \simeq M_n(\mathcal{A})$. Applying the double skew group ring construction again we conclude that

$$\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^* \simeq M_n(\mathbf{e}\mathcal{A}) \times ((1-\mathbf{e})\mathcal{A})^n \simeq \mathcal{A} \times \mathcal{A} \simeq \mathcal{A}.$$

5. Infinite partial group actions

Following Quinn [6] we define Φ in case of G being infinite as a map from $\mathcal{A} *_\alpha G$ to the ring of row and column finite matrices. Let $M_G(\mathcal{A})$ be the subring of $\mathrm{End}_k(\mathcal{A}^{|G|})$ consisting of row and column finite matrices $(a_{g,h})_{g,h \in G}$ indexed by elements of G with entries in $\mathcal{A}$, i.e. for any $g \in G$ the sets $\{a_{gh} | h \in G\}$ and $\{a_{hg} | h \in G\}$ are finite. Let $E_{g,h}$ be, as above, those matrices that are 1 in the (g,h)th component and zero elsewhere. Note that $E_{g,h}E_{r,s} = \delta_{h,r}E_{g,s}$. Then define $\Phi : \mathcal{A} *_\alpha G \to M_G(\mathcal{A})$ by

$$a\overline{g} \mapsto \sum_{h \in G} h^{-1} \cdot (g^{-1} \cdot a) E_{gh,h}$$

for any $a\overline{g} \in \mathcal{A} *_\alpha G$. Note that the (infinite) sum on the right side makes sense in $M_G(\mathcal{A})$. As above one checks that Φ is an algebra homomorphism.

Proposition 5.1. *Let G be any group acting partially on $\mathcal{A}$. Then $\mathcal{A} *_\alpha G$ is isomorphic to a subalgebra of $\mathbf{e}M_G(A)\mathbf{e}$ where $M_G(\mathcal{A})$ denotes the ring of row and column finite matrices indexed by elements of G and with entries in $\mathcal{A}$. The element $\mathbf{e}$ is the idempotent $\sum_{g\in G} 1_{g^{-1}} E_{g,g}$.*

Proof. For all $a\overline{g}, b\overline{h} \in \mathcal{A} *_\alpha G$ we have using equation (1) in the 4th line:

$$\begin{aligned}
\Phi(a\overline{g})\Phi(b\overline{h}) &= \left(\sum_{k\in G} k^{-1}\cdot(g^{-1}\cdot a)E_{gk,k}\right)\left(\sum_{l\in G} l^{-1}\cdot(h^{-1}\cdot b)E_{hl,l}\right)\\
&= \sum_{k,l\in G}(k^{-1}\cdot(g^{-1}\cdot a))(l^{-1}\cdot(h^{-1}\cdot b))E_{gk,k}E_{hl,l}\\
&= \sum_{l\in G}((hl)^{-1}\cdot(g^{-1}\cdot a))(l^{-1}\cdot(h^{-1}\cdot b))E_{ghl,l}\\
&= \sum_{l\in G} l^{-1}\cdot((gh)^{-1}\cdot(a(g\cdot b)))E_{ghl,l}\\
&= \Phi(a(g\cdot b)\overline{g}h)\\
&= \Phi((a\overline{g})(b\overline{h}))
\end{aligned}$$

Hence Φ is an algebra homomorphism. Since

$$\Phi(a\overline{g}) = 0 \Leftrightarrow (\forall h\in G): h^{-1}\cdot(g^{-1}\cdot a) = 0 \Rightarrow g\cdot(g^{-1}\cdot a) = a1_g = 0 \Rightarrow a = 0,$$

we have that Φ is injective. Moreover $\Phi(a\overline{g}) \in \mathbf{e}M_G(A)\mathbf{e}$ as above. $\square$

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